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# A predetermined algorithm for detecting a counterfeit coin with a multi-arms balance

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## Abstract

We consider the classical problem of searching a light coin in a set of  $n$  coins,  $n - 1$  of which have the same weight. The weighing device is a balance scale with  $r \geq 2$  pans that, when  $r$  equally sized subset of coins are weighted, indicates the eventual subset containing the light coin. We give a predetermined algorithm that requires the minimum possible average number of weighings for almost all values of  $n$ . © 1998 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

In this paper we consider the problem of searching for a lighter coin in a set of  $n$  coins,  $n - 1$  of which have the same weight. This is a classical problem in the area of Combinatorial Search Theory and has received considerable attention (see [1, 13] and references therein). Almost all previous papers have considered the problem when the testing device is a two-arms balance scale which compares the weights of two equally sized subsets of coins. If the weighed subsets happen to have different weights then the defective coin is known to be contained in the lighter subset. The general case of a testing device that can weigh in parallel  $r$  equally sized subsets of coins ( $r$  is an arbitrary but fixed integer greater than or equal to two), has remained open and only recently a sequential algorithm requiring the minimum average number of weighings has been given [5]. We recall that sequential algorithms are in general more powerful in that they allow the choice of the subsets weighed at the  $i$ th step to depend upon the feedbacks (outcomes) of the previous  $i - 1$  weighings while in predetermined algorithms the weighings are fixed beforehand.

In this paper we analyze the much harder case of predetermined algorithms. We consider the standard predetermined model defined in combinatorial search literature.

The reader is referred to [1, 6] for an extensive treatment of predetermined algorithms and their motivations.

The main difficulty consists in deriving the right lower bound to the minimum average number of required tests. The information theoretic lower bound here is useless in that it gives the same results both for predetermined algorithms and for the sequential algorithms. Therefore, a detailed case analysis seems necessary in order to establish non-existential results. A measure of the efficiency of the algorithm is either expressed in terms of the worst-case number of weighings or in terms of the average number of weighings required to locate the counterfeit coin. The analysis of the average-case assumes that one is given a probability distribution  $\mathbf{p} = (p_1, \dots, p_n)$ , with  $p_i$  being the probability of the  $i$ th coin being the counterfeit. In this paper we will be concerned with finding an average-case optimal algorithm under the hypothesis of uniform probability distribution on the coins' set.

In Section 3 we present a predetermined algorithm to locate a counterfeit coin that requires the minimum average number of weighings, for almost all values of  $n$ .

## 2. Preliminary results

As we already said, an algorithm is called predetermined if the sequence of tests performed by the algorithm is fixed beforehand and does not depend on the feedbacks of the tests themselves. Recall that our search model uses a balance with a number  $r$  of arms and a weighing consists of placing  $r$  equally sized subsets of coins on the  $r$  pans of the balance. Following [1, 15], we represent a predetermined algorithm on  $n$  coins by means of an  $n$ -column table. The entry  $(i, j)$  of  $\mathbf{M}$  is denoted with  $\mathbf{M}(i, j)$ . It is  $\mathbf{M}(i, j) = s$ , with  $s \in \{1, \dots, r\}$ , if the algorithm places the  $j$ th coin on pan  $s$  during the  $i$ th test.  $\mathbf{M}(i, j) = 0$  indicates that the  $j$ th coin is not placed on any pan during the  $i$ th test. Let  $A_{i,s}$  denote the set of coins which are placed on pan  $s$  at the  $i$ th weighing, i.e.  $A_{i,s} = \{j : 1 \leq j \leq n, \mathbf{M}(i, j) = s\}$ , for  $s = 1, \dots, r$ . The  $i$ th weighing compares the weights of  $r$  equal sized sets,  $A_{i,1}, \dots, A_{i,r}$ , and they are denoted by  $A_{i,1} : \dots : A_{i,r}$ . The  $i$ th weighing receives feedback  $s$ ;  $s = 0$  if all weighted subsets have equal weight, i.e. the counterfeit coin is in none of the weighed subsets, and  $1 \leq s \leq r$  if  $A_{i,s}$  is lighter than the other weighed subsets, i.e.  $A_{i,s}$  contains the counterfeit coin. Let  $l_j$  denote the number of weighings  $A_{i,1} : \dots : A_{i,r}$  which uniquely identify the counterfeit coin when the counterfeit coin is the  $j$ th one and let  $L = \max_{1 \leq j \leq n} \{l_j\}$ . Obviously, the predetermined algorithm has to specify the sets  $A_{i,1}, \dots, A_{i,r}$ , for any  $i = 1, \dots, L$ , once and for all. The counterfeit coin is the  $j$ th one only if the feedback to the  $i$ th weighing is equal to  $\mathbf{M}(i, j)$ , for  $i = 1, \dots, L$ . Since the  $j$ th coin is uniquely identified after  $l_j$  weighings, we can cross out the last  $L - l_j$  entries in the  $j$ th column of  $\mathbf{M}$ . As a consequence, the table representing the predetermined algorithm may have columns of different lengths. Consider the predetermined algorithm that at the generic step  $i$  performs the weighing  $A_{i,1} : \dots : A_{i,r}$ . If the table representing the algorithm contains

no pair of columns such that one is prefix of the other, then the counterfeit coin is uniquely identified as soon as the sequence of feedbacks coincides with a column of  $\mathbf{M}$ . The index of this column corresponds to the index of the counterfeit coin. For a more pedagogic introduction to predetermined algorithms see Section 1.8 of [1].

The following result establishes the conditions for a table to represent an algorithm which finds a counterfeit coin in a set of  $n$  coins under our search model.

**Lemma 1.** *Let  $n \geq r$ . An  $n$ -column table  $\mathbf{M}$  with entries in  $\{0, 1, \dots, r\}$  represents a predetermined algorithm if and only if the following conditions hold:*

(a) *Let  $n$  be the number of columns of  $\mathbf{M}$ , for each row  $i$*

$$|\{j : 1 \leq j \leq n, \mathbf{M}(i, j) = 1\}| = \dots = |\{j : 1 \leq j \leq n, \mathbf{M}(i, j) = r\}|.$$

(b) *For each pair  $\mathbf{M}_i$  and  $\mathbf{M}_j$  of columns of  $\mathbf{M}$ ,  $\mathbf{M}_i$  is not a prefix of  $\mathbf{M}_j$ .*

**Proof.** The proof that conditions (a) and (b) imply an algorithm which finds the counterfeit coin follows from the above discussion. The necessity of (a) is immediate since each weighing needs to place the same number of coins on each pan. The necessity of (b) follows from the fact that if  $\mathbf{M}_i$  were a prefix of  $\mathbf{M}_j$  then the algorithm could not decide whether the counterfeit coin is the one associated with column  $\mathbf{M}_i$  or the one associated with column  $\mathbf{M}_j$  whenever it gets a sequence of feedbacks equal to  $\mathbf{M}_i$ .  $\square$

A table which satisfies the conditions of Lemma 1 is said *admissible*.

If we are concerned with finding an optimal *worst-case* algorithm, our problem consists of constructing an admissible table which minimizes the length of the longest column. To this aim, we can restrict our analysis to admissible tables which have columns of the same length. Among such tables we find the one which has the smallest number of rows. Notice that in the case of a table with all columns of the same length, property (b) of Lemma 1 implies that the columns in the table are pairwise distinct. The following lemma establishes that an optimal worst-case predetermined algorithm finds the counterfeit coin in  $\lceil \log_{r+1} n \rceil + 1$  tests if  $n$  is of the form  $n = (r+1)^L - i$ , for some positive integer  $L$  and  $2 \leq i \leq r-1$ , and in  $\lceil \log_{r+1} n \rceil$  tests in all other cases.

We will denote with  $\mathbf{i}^\ell$  the  $\ell$ -entry column having all the entries equal to  $i$  for some  $i \in \{0, 1, \dots, r\}$ .

**Lemma 2.** *Given an integer  $m \geq r$  with  $(r+1)^{\ell-1} < m \leq (r+1)^\ell$ , there exists an admissible table  $\mathbf{A}$  with  $m$  columns each of length  $\ell$  if and only if either  $m \leq (r+1)^\ell - r$  or  $m = (r+1)^\ell - 1$ . Otherwise there exists an admissible table with all columns of length  $\ell+1$ . It is possible to construct  $\mathbf{A}$  in such a way that it does not contain the all-zero column if and only if either  $m \neq (r+1)^\ell - r$  and  $m \neq (r+1)^\ell$ . Moreover, it*

is possible to construct  $A$  in such a way that it does not contain any of the columns  $0^\ell, \dots, r^\ell$  if and only if  $m \leq (r+1)^\ell - 2r$  or  $m = (r+1)^\ell - r - 1$ .

**Proof.** See Appendix A.  $\square$

The following result is an easy consequence of the above lemma and will be very useful in proving the main theorem of the paper.

**Corollary 1.** *Given an integer  $m \geq r$  with  $r \leq m \leq (r+1)^\ell - r$  or  $m = (r+1)^\ell - 1$ , there exists an admissible table with  $m$  columns each of length  $\ell$ . It is possible to construct the admissible table in such a way that it does not contain the all-zero column if and only if either  $m \neq (r+1)^\ell - r$  and  $m \neq (r+1)^\ell$ . Moreover, it is possible to construct the table in such a way that it does not contain any of the columns  $0^\ell, \dots, r^\ell$  if and only if  $m \leq (r+1)^\ell - 2r$  or  $m = (r+1)^\ell - r - 1$ .*

**Proof.** Let  $M$  be the admissible table of size  $\ell' \times m$  from Lemma 2. If  $\ell' = \ell$  the new table is  $M$  itself. If  $\ell' < \ell$  we construct a table  $M'$  coinciding with  $M$  in the first  $\ell'$  rows and with any table which satisfies property (a) of Lemma 1 in the last  $\ell - \ell'$  rows. Notice that table  $M'$  satisfies property (b) of Lemma 1 as well, in that the columns of table  $M$  are pairwise distinct.  $\square$

Let us now consider the problem of determining the minimum average number of weighings under the assumption of uniform distribution on the search set.

Recall that the columns of an admissible table  $M$  may have different lengths and that  $l_j$  denotes the length of the  $j$ th column of  $M$ . For all  $i > j$ , entry  $M(i, j)$  is empty and indicates that the  $j$ th coin is known to be counterfeit in less than  $i$  weighings. The average number of weighings of a predetermined algorithm described by a table  $M$  corresponds then to the number  $W(M)$  of non-empty entries of  $M$  divided by the number  $n$  of columns of  $M$ .

We want to point out an important correspondence existing between tables which satisfy property (b) of Lemma 1 and  $(r+1)$ -ary trees. Let  $M$  be an  $n$ -column table which verifies property (b) of Lemma 1. A labeled  $(r+1)$ -ary tree is an  $(r+1)$ -ary tree such that the edges connecting an internal node to its children are each labeled with a distinct integer in the set  $\{0, 1, \dots, r\}$ . The tree  $T$  associated with  $M$  is a labeled  $(r+1)$ -ary tree with  $n$  leaves and such that each root-to-leaf path is associated to a distinct column of  $M$ . The lengths of the root-to-leaf paths of  $T$  are the lengths of the columns in  $M$  and the labels along each root-to-leaf path (reading from the root to the leaf) coincide with the entries of a distinct column of  $M$  (reading from the top to the bottom).

Let  $T$  be any  $(r+1)$ -ary tree. The sum of the lengths of the paths from the root to the leaves is called external length of  $T$  and is denoted with  $h(T)$ . If  $T$  represents an admissible table  $M$  then it is obvious that  $W(M) = h(T)$ . Let  $H(n) = \min h(T)$ , where the minimum is taken over all  $(r+1)$ -ary trees with  $n$  leaves.

The following well-known result (e.g., see [1]) allows to find  $H(n)$  explicitly.

**Theorem 1.** *Given an integer  $n, n = (r+1)^L + kr + j$ , where  $0 \leq k < (r+1)^L$ ,  $0 \leq j \leq r-1$ , an  $n$ -node tree  $T$  has external path length  $h(T)$  equal to  $H(n)$  if and only if  $T$  has  $n - \lceil (kr+j)(r+1)/r \rceil$  leaves at level  $L$  and  $\lceil (kr+j)(r+1)/r \rceil$  at level  $L+1$ . Moreover,*

$$H(n) = nL + \left\lceil (kr+j) \frac{r+1}{r} \right\rceil = n \lfloor \log_{r+1} n \rfloor + \left\lceil \left( n - (r+1)^{\lfloor \log_{r+1} n \rfloor} \right) \frac{r+1}{r} \right\rceil.$$

Let  $T_L$  be the tree with  $(r+1)^L$  leaves at level  $L$ . A tree with  $n$  leaves and having external path length equal to  $H(n)$  can be obtained from  $T_L$  by changing  $k$  leaves into internal nodes with  $r+1$  sons each if  $j=0$  and, if  $j>0$ , one more leaf into an internal node with  $j+1$  sons.

Since we are concerned with admissible tables only, we consider  $(r+1)$ -ary labeled trees which correspond to tables which satisfy property (a) of Lemma 1. Therefore, our problem consists of minimizing  $h(T)$  over this restricted class of  $(r+1)$ -ary trees. We just point out that our problem is considerably harder than minimizing  $h(T)$  in the unrestricted case. The following corollary of Theorem 1 is an immediate consequence of the above discussion.

**Corollary 2.** *Let  $M$  be an  $n$ -column admissible table. One has:*

$$W(M) \geq H(n).$$

### 3. An almost optimal average case algorithm

The main result of this section is the following theorem.

**Theorem 2.** *Let  $n = (r+1)^L + kr + j$ , for some integers  $L \geq 1, 0 \leq k < (r+1)^L$  and  $0 \leq j \leq r-1$ , and define*

$$\alpha(j) = \begin{cases} 0 & \text{if } j = 0, \\ 1 & \text{if } j > 0. \end{cases} \quad (1)$$

*The minimum average number of weighings  $\bar{L}_{\text{pre}}(n)$  done by any predetermined algorithm that locates a counterfeit coin out of a set of  $n$  coins is equal to:*

$$\bar{L}_{\text{pre}}(n) = \frac{H(n)}{n} + \left\{ \begin{array}{ll} 0 & \begin{array}{l} \text{if } j = 0 \text{ and } k = 0 \\ \text{or if } j \in \{0, r-1\} \text{ and } ((k = 1 - \alpha(j)) \\ \text{or } (r \leq k \leq (r+1)^L - r - \alpha(j), L \geq 2) \\ \text{or } (k = (r+1)^L - 1)), \end{array} \\ \\ \frac{1}{n} & \begin{array}{l} \text{if } k = (r+1)^L - 2 \\ \text{or if } 1 \leq j \leq r-2 \text{ and } r \leq k \leq 2r-j-1 \\ \text{or if } 2 \leq j \leq r-2 \text{ and} \\ \quad 2r-j \leq k \leq 2r-2 \text{ and } L \geq r-j-2 \\ \text{or if } 1 \leq j \leq r-2 \text{ and} \\ \quad 2r-1 \leq k \leq (r+1)^L - r - 1, L \geq 2, \end{array} \\ \\ \frac{2}{n} & \begin{array}{l} \text{if } 2 \leq j \leq r-2 \text{ and} \\ \quad 2r-j \leq k \leq 2r-2 \\ \text{and } L \leq r-j-3, \end{array} \\ \\ \frac{r-j-1}{n} & \text{if } 1 \leq j \leq r-2 \text{ and } k = 0, \\ \\ d_1 \leq \frac{r-k}{n} & \begin{array}{l} \text{if } 2 - \alpha(j) \leq k \leq r-1, \\ \text{with the equality holding for} \\ 2(r+1) \leq n \leq r(r+1) \text{ and for} \\ (r+1)^L + r^2 - r \leq n \leq (r+1)^L + r^2 - 1, \end{array} \\ \\ \frac{(r+1)^L - k - 1 - \alpha(j)}{n} & \begin{array}{l} \text{if } (r+1)^L - r + 1 - \alpha(j) \leq k \leq (r+1)^L - 3 \\ \text{and } L \geq 2, \end{array} \\ \\ d_2 \leq \frac{r}{n} & \begin{array}{l} \text{if } 1 \leq j \leq r-2 \text{ and } k = (r+1)^L - 1 \\ \text{and } L \geq 2. \end{array} \end{array} \right.$$

Given a table  $\mathbf{M}$  with  $n$  columns define  $\Delta(\mathbf{M}) = W(\mathbf{M}) - H(n)$ . By Theorem 1, one has that  $H(n) = nL + k(r+1) + j + \alpha(j)$ . Moreover, define

$$\Delta(n) = \min \Delta(\mathbf{M}) \quad (2)$$

where the minimum is taken over all admissible tables with  $n$  columns.

Next lemma provides an upper bound to  $\Delta(n)$  and, consequently, yields an upper bound to  $\bar{L}_{\text{pre}}(n)$ .

**Lemma 3.** Let  $n = (r+1)^L + kr + j$ , for some integers  $L \geq 1, 0 \leq k < (r+1)^L$  and  $0 \leq j \leq r-1$ . One has

$$\Delta(n) \leq \left\{ \begin{array}{ll}
 0 & \begin{array}{l} \text{if } j = 0 \text{ and } k = 0 \\ \text{or if } j \in \{0, r-1\} \text{ and } ((k = 1 - \alpha(j)) \\ \text{or } (r \leq k \leq (r+1)^L - r - \alpha(j), L \geq 2) \\ \text{or } (k = (r+1)^L - 1)), \end{array} \\
 1 & \begin{array}{l} \text{if } k = (r+1)^L - 2 \\ \text{or if } 1 \leq j \leq r-2 \text{ and } r \leq k \leq 2r-j-1 \\ \text{or if } 2 \leq j \leq r-2 \text{ and } 2r-j \leq k \leq 2r-2 \\ \text{and } L \geq r-j-2 \\ \text{or if } 1 \leq j \leq r-2 \text{ and } \\ 2r-1 \leq k \leq (r+1)^L - r - 1, L \geq 2 \end{array} \\
 2 & \begin{array}{l} \text{if } 2 \leq j \leq r-2, 2r-j \leq k \leq 2r-2 \\ \text{and } L \leq r-j-3, \end{array} \\
 r-j-1 & \text{if } 1 \leq j \leq r-2 \text{ and } k = 0, \\
 r-k & \text{if } 2 - \alpha(j) \leq k \leq r-1, \\
 (r+1)^L - k - 1 - \alpha(j) & \text{if } (r+1)^L - r + 1 - \alpha(j) \leq k \leq (r+1)^L - 3 \\ & \text{and } L \geq 2, \\
 r & \text{if } 1 \leq j \leq r-2 \text{ and } k = (r+1)^L - 1 \\ & \text{and } L \geq 2. \end{array} \right.$$

where  $\alpha(j)$  is defined in (1).

To prove the lemma, we establish the following conventions in order to describe the tables we are going to construct.

**Notations.** The following list summarize the basic notation.

- For each integer  $i$ ,  $\ell$ , and  $m$  we denote by  $i^{(\ell \times m)}$  the table with  $\ell$  rows and  $m$  columns having each entry equal to  $i$ . We denote  $i^{(\ell \times 1)}$  by  $i^{(\ell)}$ .
- Given two integers  $i$  and  $j$  with  $i \leq j$ , the notation  $i \dots j$  indicates  $j-i+1$  consecutive row entries containing the integers from  $i$  to  $j$ . If  $i > j$  then  $i \dots j$  specifies no row entry.
- We denote by  $0 \dots 0$  a sequence of row entries each of which is equal to zero. The number of such entries will be clear from the context.
- Given two tables  $M$  and  $M'$ , we denote by  $[M|M']$  the table whose first columns are those of  $M$  and the last ones are those of  $M'$ .

- Given a table  $M$  with all columns of the same length and a row vector  $v$ , we denote by

$$\begin{bmatrix} M \\ v \end{bmatrix}$$

the table obtained from  $M$  by adding a last row equal to vector  $v$ .

- We denote by  $|M|$  the number of columns of the table  $M$  and by  $M \subseteq M'$  we mean that each column of  $M$  is also a column of  $M'$ .
- If  $M \subseteq M'$  we denote by  $M' - M$  the set of columns obtained by eliminating the columns of  $M$  from the columns' set of  $M'$ .
- We denote by  $M_L$  a table whose columns are all vectors of length  $L$  on  $\{0, 1, \dots, r\}$ . Trivially,  $M_L$  is admissible and  $W(M_L) = H((r+1)^L)$ .

**Proof of Lemma 3.** Let  $n = (r+1)^L + kr + j$ , for some integers  $L \geq 1, 0 \leq k < (r+1)^L$  and  $0 \leq j \leq r-1$ . We show that there exists a table  $M$  with  $n$  columns such that  $\Delta(M)$  satisfies the desired upper bound.

We distinguish different cases depending on the values of  $k$  and  $j$ .

*Case 1:*  $2 - \alpha(j) \leq k \leq r-1$ . Let  $H$  be the  $L \times r$  table  $H = [1^{(L)} | 2^{(L)} | \dots | r^{(L)}]$ . We can then construct the admissible table

$$M = \begin{cases} \left[ \begin{array}{c|c|c|c|c} M_L - H & H & H & \dots & H \\ 12 \dots r & 2 \dots r1 & & & (k+1) \dots r1 \dots k \end{array} \right] & \text{if } j = 0, \\ \\ \left[ \begin{array}{c|c|c|c|c} M_L - H - 0^{(L)} & H & & & H \\ 12 \dots r & & & & k \dots r1 \dots (k-1) \end{array} \right] \\ \left[ \begin{array}{c|c} H & 0^{(L \times (j+1))} \\ (k+1) \dots (k+r-j-1) 0 \dots 0 & (k+r-j) \dots r1 \dots k \end{array} \right] & \text{if } j > 0, k \leq j+1, \\ \\ \left[ \begin{array}{c|c|c|c|c} M_L - H - 0^{(L)} & H & & & H \\ 12 \dots r & & & & k \dots r1 \dots (k-1) \end{array} \right] \\ \left[ \begin{array}{c|c} H & 0^{(L \times (j+1))} \\ (k+1) \dots r1 \dots (k-j-1) 0 \dots 0 & (k-j) \dots k \end{array} \right] & \text{if } j > 0, k > j+1 \end{cases}$$

with the number of non-empty entries equal to

$$W(M) = nL + (k+1)r + (j + \alpha(j)) = H(n) + (r-k).$$

*Case 2:*  $k = 1 - \alpha(j)$ . We can construct the admissible table



$$M = \begin{cases} \left[ M_L - \mathbf{0}^{(L)} \mid \begin{smallmatrix} \mathbf{0}^{(L)} \\ 0 \end{smallmatrix} \mid \begin{smallmatrix} \mathbf{0}^{(L)} \\ 1 \end{smallmatrix} \mid \cdots \mid \begin{smallmatrix} \mathbf{0}^{(L)} \\ r \end{smallmatrix} \right] & \text{if } j = 0, \\ \left[ M_L - \mathbf{0}^{(L)} - c_1 - \cdots - c_{r-j-1} \mid \begin{smallmatrix} c_1 \\ 1 \end{smallmatrix} \mid \cdots \mid \begin{smallmatrix} c_{r-j-1} \\ r-j-1 \end{smallmatrix} \mid \begin{smallmatrix} \mathbf{0}^{(L \times (j+1))} \\ (r-j) \cdots r \end{smallmatrix} \right] & \text{if } j > 0 \end{cases}$$

where  $c_1, \dots, c_{r-j-1}$  are any columns of  $M_L - \mathbf{0}^{(L)}$ . The number of non-empty entries of  $M$  is

$$W(M) = nL + \begin{cases} r+1 & \text{if } j = 0 \\ r & \text{if } j > 0 \end{cases} = H(n) + \begin{cases} 0 & \text{if } j = 0, \\ r-j-1 & \text{if } 1 \leq j \leq r-1. \end{cases}$$

Case 3:  $L \geq 2$  and  $r \leq k \leq (r+1)^L - r - \alpha(j)$ .

Case 3.1:  $j = 0$  or  $j = r-1$ .

Let  $H$  be the  $L \times k$  admissible table from Corollary 1. We can then construct the admissible table

$$M = \begin{cases} \left[ M_L - H \mid \begin{smallmatrix} H \\ \mathbf{0}^{(1 \times k)} \end{smallmatrix} \mid \begin{smallmatrix} H \\ \mathbf{1}^{(1 \times k)} \end{smallmatrix} \mid \cdots \mid \begin{smallmatrix} H \\ r^{(1 \times k)} \end{smallmatrix} \right] & \text{if } j = 0, \\ \left[ M_L - H - \mathbf{0}^{(L)} \mid \begin{smallmatrix} H \\ \mathbf{0}^{(1 \times k)} \end{smallmatrix} \mid \cdots \mid \begin{smallmatrix} H \\ r^{(1 \times k)} \end{smallmatrix} \mid \begin{smallmatrix} \mathbf{0}^{(L \times r)} \\ 1 \cdots r \end{smallmatrix} \right] & \text{if } j = r-1. \end{cases}$$

The number of non-empty entries of the given matrices is

$$W(M) = H(n) = nL + k(r+1) + \begin{cases} 0 & \text{if } j = 0, \\ r & \text{if } j = r-1. \end{cases}$$

Case 3.2:  $1 \leq j \leq r-2$ .

- If  $r \leq k \leq 2r-j-1$ , let  $H$  be an  $(L \times (k+1))$  admissible table not containing the all-zero column from Corollary 1 and for  $i = 1, \dots, r$ , let  $v_i = v_{i,1} \cdots, v_{i,k+1}$  denote the  $(1 \times (k+1))$  vector defined by

$$v_{i,t} = \begin{cases} i & \text{if } t \neq i, \\ 0 & \text{if } t = i. \end{cases} \quad (3)$$

We can construct the admissible table

$$M = \left[ M_L - H - \mathbf{0}^{(L)} \mid \begin{smallmatrix} H \\ v_1 \end{smallmatrix} \mid \cdots \mid \begin{smallmatrix} H \\ v_{k-r+j+1} \end{smallmatrix} \mid \begin{smallmatrix} H \\ (k-r+j+2)^{(1 \times (k+1))} \end{smallmatrix} \right] \\ \cdots \left[ \begin{smallmatrix} H \\ r^{(1 \times (k+1))} \end{smallmatrix} \mid \begin{smallmatrix} \mathbf{0}^{(L \times (k-r+j+2))} \\ 0 \cdots (k-r+j+1) \end{smallmatrix} \right]$$

with number of non-empty entries equal to

$$W(M) = nL + r(k+1) + k - r + j + 2 = H(n) + 1.$$

- If  $2r - j \leq k \leq 2r - 2$  ( $j \geq 2$ ) and  $L \leq r - j - 3$ , let  $H$  be an  $(L \times (k+2))$  admissible table not containing the all-zero column from Corollary 1. We can construct the admissible table

$$M = \left[ \begin{array}{c|c|c|c} M_L - H - \mathbf{0}^{(L)} & H & \dots & H \\ \hline v_1 & & v_{k-2r+j+3} & (k-2r+j+4)^{(1 \times (k+2))} \end{array} \right] \\ \left[ \begin{array}{c|c} H & \mathbf{0}^{(L \times (k-2r+j+3))} \\ \hline r^{(1 \times (k+2))} & 1 \dots (k-2r+j+3) \end{array} \right]$$

where the  $v_i$ 's are defined in (3). Then it results

$$W(M) = nL + (k+2)r + (k-2r+j+3) = nL + k(r+1) + j + 3 = H(n) + 2.$$

- If  $2r - j \leq k \leq 2r - 2$  ( $j \geq 2$ ) and  $L \geq r - j - 2$ , let  $d_i = (d_{i,1}, \dots, d_{i,L})$  be the column vector of length  $L$  defined as follows:

$$d_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (4)$$

$M$  can be written as

$$M = \left[ M_L - H \mid \left[ \begin{array}{c|c|c|c} H & H & \dots & H \\ \hline \mathbf{0}^{(1 \times |H|)} & \mathbf{1}^{(1 \times |H|)} & \dots & r^{(1 \times |H|)} \end{array} \right] - B \right]$$

where  $H$  is an  $L \times (k+2)$  table and  $B$  is an  $(L+1) \times (2r-j)$  table. If  $j < r-3$ , they are defined as follows:

$$H = \left[ \begin{array}{c|c|c} d_1 \dots d_{r-j-2} & \mathbf{1}^{(r-j-2)} \dots (h+1)^{(r-j-2)} & (h+2)^{(r-j-3)} \dots r^{(r-j-3)} \\ \hline & \mathbf{0}^{((L-r+j+2) \times (h+1))} & \mathbf{0}^{((L-r+j+3) \times (r-h-1))} \end{array} \right] \\ \left[ \begin{array}{c} \mathbf{0}^{((r-j-3) \times (r-h))} \\ (h+2) \dots r \ 0 \\ \hline \mathbf{0}^{((L-r+j+2) \times (r-h))} \end{array} \right]$$

with  $h = 3r - j - k - 4$ .

$$B = \left[ \begin{array}{c|c|c|c} d_1 \dots d_{r-j-2} & \mathbf{1}^{(r-j-2)} & \mathbf{1}^{(r-j-2)} & \dots \\ \hline \mathbf{0}^{(1 \times (r-j-2))} & \mathbf{0}^{(L-r+j+2)} & \mathbf{0}^{(L-r+j+2)} & \dots \\ & 0 & 1 & \dots \end{array} \right] \\ \left[ \begin{array}{c|c|c|c} \mathbf{1}^{(r-j-2)} & \mathbf{0}^{(r-j-2)} & \mathbf{0}^{(r-j-2)} & \mathbf{0}^{(L+1)} \\ \hline \mathbf{0}^{(L-r+j+2)} & \mathbf{0}^{(L-r+j+2)} & \mathbf{0}^{(L-r+j+2)} & \mathbf{0}^{(L+1)} \\ & r-2 & r-1 & r \end{array} \right].$$

For  $j \in \{r-3, r-2\}$ , the constructions for  $\mathbf{H}$  and  $\mathbf{B}$  are similar to those given above, and can be easily obtained.

It is  $W(\mathbf{M}) = nL + k(r+1) + j + 2 = H(n) + 1$ .

- If  $2r-1 \leq k \leq (r+1)^L - r - 2$ , let  $\mathbf{A}$  be an admissible table of size  $L \times ((r+1)^L - k - 2)$ , not containing the columns  $\mathbf{0}^{(L)}, \mathbf{1}^{(L)}, \dots, \mathbf{r}^{(L)}$ . Since  $r \leq |\mathbf{A}| \leq (r+1)^L - 2r - 1$ , Corollary 1 assures the existence of such a table. Defining  $\mathbf{B}$  as

$$\mathbf{B} = \begin{bmatrix} \mathbf{0}^{(L \times r-j)} \mathbf{1}^{(L)} \dots \mathbf{r}^{(L)} \\ 1 \dots r \ 0 \ 0 \dots 0 \end{bmatrix}$$

the desired table is

$$\mathbf{M} = \left[ \mathbf{A} \mid \mathbf{M}_{L+1} - \begin{bmatrix} \mathbf{A} & \mathbf{A} & \dots & \mathbf{A} \\ \mathbf{0}^{(1 \times |\mathbf{A}|)} & \mathbf{1}^{(1 \times |\mathbf{A}|)} & \dots & \mathbf{r}^{(1 \times |\mathbf{A}|)} \end{bmatrix} - \mathbf{B} \right]$$

with

$$W(\mathbf{M}) = nL + k(r+1) + j + 2 = H(n) + 1.$$

- If  $k = (r+1)^L - r - 1$ , let  $\mathbf{A}$  and  $\mathbf{B}$  be respectively, the  $L \times (r-1)$  table and the  $3 \times (2r-j)$  table defined as follows:

$$\mathbf{A} = \begin{bmatrix} 2 \dots r \\ \mathbf{0}^{((L-1) \times (r-1))} \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 1 & \dots & 1 & & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & (r-j+1) & \dots & r & 1 & \dots & (r-j) \\ 1 & \dots & (r-j) & (r-j+1) & \dots & r & 0 & \dots & 0 \end{bmatrix}.$$

We can then construct the admissible table

$$\mathbf{M} = \left[ \mathbf{A} \mid \mathbf{M}_{L+1} - \begin{bmatrix} \mathbf{A} & \mathbf{A} & \dots & \mathbf{A} \\ \mathbf{0}^{(1 \times |\mathbf{A}|)} & \mathbf{1}^{(1 \times |\mathbf{A}|)} & \dots & \mathbf{r}^{(1 \times |\mathbf{A}|)} \end{bmatrix} - \begin{bmatrix} \mathbf{B} \\ \mathbf{0}^{((L-2) \times (2r-j))} \end{bmatrix} \right].$$

The number of the non-empty entries in the given matrix is

$$W(\mathbf{M}) = n(L+1) - r + 1 = nL + (r+1)^{L+1} - r^2 - 2r + j + 1 = H(n) + 1.$$

*Case 4:*  $(r+1)^L - r + 1 - \alpha(j) \leq k \leq (r+1)^L - 3$  and  $L \geq 2$ . Let  $i = (r+1)^L - k$ . From Corollary 1, there exists an  $L \times ((i-1)r-j)$  admissible table  $\mathbf{H}$  not containing the all-zero column. Let  $\mathbf{v}$  be any row of length  $(i-1)r-j$  which satisfies condition (a) of Lemma 1. We construct the admissible table

$$\mathbf{M} = \left[ \mathbf{0}^{(L)} \mid \mathbf{M}_{L+1} - \begin{bmatrix} \mathbf{H} \\ \mathbf{v} \end{bmatrix} - \begin{bmatrix} \mathbf{0}^{(L \times (r+1))} \\ 0 \ 1 \dots r \end{bmatrix} \right]$$

with number of non-empty entries

$$W(\mathbf{M}) = nL + n - 1 = H(n) + (r+1)^L - k - 1 - \alpha(j).$$

*Case 5:*  $L \geq 2$  and  $k = (r+1)^L - 2$

- If  $j = 0$ , let  $H$  denote the table  $[1^{(L+1)} | 2^{(L+1)} | \dots | r^{(L+1)}]$ . We construct the admissible table

$$M = \left[ \begin{array}{c} \mathbf{0}^{(L)} \\ M_{L+1} - H - \left[ \begin{array}{c} \mathbf{0}^{(L \times (r+1))} \\ 01 \dots r \end{array} \right] \end{array} \right].$$

- If  $j > 0$ ,  $M$  is the  $(L+1) \times n$  table of Lemma 1.

Then

$$W(M) = H(n) + 1 = \begin{cases} n(L+1) - 1 & \text{if } j = 0, \\ n(L+1) & \text{if } j > 0. \end{cases}$$

Case 6:  $k = (r+1)^L - 1$

- If  $j = 0$ , we construct

$$M = \left[ \begin{array}{c} \mathbf{0}^{(L)} \\ M_{L+1} - \left[ \begin{array}{c} \mathbf{0}^{(L \times (r+1))} \\ 0 \ 1 \ \dots \ r \end{array} \right] \end{array} \right]$$

- If  $j = r-1$ , we set  $M = M_{L+1} - [\mathbf{0}^{(L+1)}]$ .
- If  $1 \leq j \leq r-2$ , let  $H = [1^{(L+1)} | 2^{(L+1)} | \dots | r^{(L+1)}]$  and  $c_1, \dots, c_{r-j-1}$  be any non-zero columns of  $M_{L+1} - H$ . We construct the admissible table

$$M = \left[ \begin{array}{c} M_{L+1} - H - \mathbf{0}^{(L+1)} - c_1 - \dots - c_{r-j-1} \\ \left[ \begin{array}{c} c_1 \\ 1 \end{array} \right] \quad \dots \quad \left[ \begin{array}{c} c_{r-j-1} \\ r-j-1 \end{array} \right] \quad \left[ \begin{array}{c} \mathbf{0}^{((L+1) \times (j+1))} \\ (r-j) \ \dots \ r \end{array} \right] \end{array} \right].$$

The number of non-empty entries of the given tables is

$$W(M) = nL + \begin{cases} n-1 & \text{if } j = 0, \\ n & \text{if } j = r-1, \\ n+r & \text{if } 1 \leq j \leq r-2, \end{cases} = H(n) + \begin{cases} 0 & \text{if } j = 0 \text{ or } j = r-1, \\ r & \text{if } 1 \leq j \leq r-2. \end{cases} \quad \square$$

The above lemma provides the upper bound to  $\bar{L}_{\text{pre}}(n)$  stated in Theorem 2. Now we turn our attention to the lower bound stated in that theorem. We say that an admissible table  $M$  with  $n$  columns is *optimal* if it has the minimum number of non-empty entries among all admissible tables with  $n$  columns, i.e. if  $\Delta(M) = \Delta(n)$ .

**Lemma 4.** *At least one optimal table has column lengths differing by at most 1, with the exception of the case when  $n$  is of the form  $n = (r+1)^L - r + j$ , for some integers  $L \geq 1$  and  $1 \leq j \leq r-2$ .*

**Proof.** Let  $M$  be an optimal admissible table with maximum column length  $\ell_{\max}$ . From the optimality of  $M$ , it follows that there are non-empty entries of the last row of  $M$  which are different from zero. If this was not the case, the table obtained by removing such zero entries would still satisfy condition (b) of Lemma 1. As a consequence, there would be an admissible table with a number of non-empty entries smaller than the number of non-empty entries of  $M$ . Condition (a) of Lemma 1 implies that for each  $s = 1, \dots, r$ ,  $M$  contains a column  $c_s$  of length  $\ell_{\max}$  with  $s$  in the last position.

Suppose now, by contradiction, that there exists a column  $c$  of  $M$  of length  $\ell \leq \ell_{\max} - 2$ . Consider the table

$$M' = \left[ M - c - c_1 - \cdots - c_r \mid \begin{array}{c} c \\ 0 \end{array} \mid \begin{array}{c} c \\ 1 \end{array} \mid \cdots \mid \begin{array}{c} c \\ r \end{array} \right]. \quad (5)$$

The table  $M'$  has the property that each column is not a prefix of any other column, therefore,  $W(M') \geq H(n)$ . On the other hand, since it is  $W(M) = W(M') + r(\ell_{\max} - \ell - 1) - 1 \geq W(M') + r - 1$ , we get  $W(M) \geq H(n) + r - 1$ . By Lemma 3, either  $M$  is not optimal or the matrix constructed in Lemma 3 (that satisfies the thesis of this lemma) is also optimal, unless  $n = (r+1)^L - r + j$ ,  $1 \leq j \leq r - 2$ .  $\square$

Lemma 3 has already given the desired upper bound to  $\Delta(n)$ , we prove now a matching lower bound thus completing the proof of Theorem 2.

**Lemma 5.** For each admissible  $n$ -column table  $M$ , with  $n = (r+1)^L + kr + j$ , for some integers  $L \geq 1$ ,  $0 \leq k < (r+1)^L$  and  $0 \leq j \leq r-1$

$$\Delta(M) \geq \left\{ \begin{array}{ll} 0 & \begin{array}{l} \text{if } j = 0 \text{ and } k = 0 \\ \text{or if } j \in \{0, r-1\} \text{ and } ((k = 1 - \alpha(j)) \\ \text{or } (r \leq k \leq (r+1)^L - r - \alpha(j), L \geq 2) \\ \text{or } (k = (r+1)^L - 1)), \end{array} \\ \\ 1 & \begin{array}{l} \text{if } 2 - \alpha(j) \leq k \leq r-1 \text{ and } L \geq 2 \\ \text{or if } k = (r+1)^L - 2 \\ \text{or if } 1 \leq j \leq r-2 \text{ and } k = (r+1)^L - 1 \\ \text{or if } 1 \leq j \leq r-2 \text{ and } r \leq k \leq 2r-j-1 \\ \text{or if } 2 \leq j \leq r-2 \text{ and } 2r-j \leq k \leq 2r-2 \\ \text{and } L \geq r-j-2 \\ \text{or if } 1 \leq j \leq r-2 \text{ and } \\ 2r-1 \leq k \leq (r+1)^L - r - 1 \text{ and } L \geq 2, \end{array} \\ \\ 2 & \begin{array}{l} \text{if } 2 \leq j \leq r-2 \text{ and } 2r-j \leq k \leq 2r-2 \\ \text{and } L \leq r-j-3, \end{array} \\ \\ r-j-1 & \text{if } 1 \leq j \leq r-2 \text{ and } k = 0, \\ \\ r-k & \text{if } 2(r+1) \leq n \leq r(r+1), \\ \\ (r+1)^L - k - 1 - \alpha(j) & \begin{array}{l} \text{if } (r+1)^L - r + 1 - \alpha(j) \leq k \leq (r+1)^L - 3 \\ \text{and } L \geq 2. \end{array} \end{array} \right.$$

**Proof.** By Lemma 4 we can limit ourselves to consider only tables with columns of length  $L$  and  $L + 1$ , or  $L + 1$  and  $L + 2$  in case  $n = (r + 1)^{L+1} - r + j$ .

Given an admissible table  $\mathbf{M}$  with  $n$  columns each of length  $L$  or  $L + 1$ , we can write it as

$$\mathbf{M} = \left[ \mathbf{A} \mid M_{L+1} - \left[ \begin{smallmatrix} \mathbf{A} \\ \mathbf{0}^{(|\mathbf{A}|)} \end{smallmatrix} \right] - \cdots - \left[ \begin{smallmatrix} \mathbf{A} \\ \mathbf{r}^{(|\mathbf{A}|)} \end{smallmatrix} \right] - \mathbf{B} \right] \quad (6)$$

for some  $\mathbf{A} \subseteq \mathbf{M}_L$  and  $\mathbf{B} \subseteq \mathbf{M}_{L+1}$  such that no column of  $\mathbf{A}$  is prefix of a column of  $\mathbf{B}$ . An easy calculation shows that

$$|\mathbf{A}| = (r + 1)^L - k - \alpha(j) - \Delta(\mathbf{M}) \quad \text{and} \quad |\mathbf{B}| = (\Delta(\mathbf{M}) + \alpha(j))r - j. \quad (7)$$

If we suppose  $\Delta(\mathbf{M}) = 0$ , we get

$$|\mathbf{A}| = (r + 1)^L - k - \alpha(j) \quad \text{and} \quad |\mathbf{B}| = r\alpha(j) - j \quad (8)$$

We notice now that if  $\mathbf{A}$  does not verify property (a) of Lemma 1 then there is some row  $t$  and some integers  $i, i' \in \{1, \dots, r\}$  such that the difference between the number of  $i$ 's and of  $i'$ 's in the row  $t$  of

$$\left[ \mathbf{A} \mid M_{L+1} - \left[ \begin{smallmatrix} \mathbf{A} \\ \mathbf{0}^{(|\mathbf{A}|)} \end{smallmatrix} \right] - \cdots - \left[ \begin{smallmatrix} \mathbf{A} \\ \mathbf{r}^{(|\mathbf{A}|)} \end{smallmatrix} \right] \right]$$

is  $hr$ , for some  $h \geq 1$ , and cannot be balanced by any choice of  $\mathbf{B}$  since  $|\mathbf{B}| = r\alpha(j) - j < r$ . Therefore,  $\mathbf{A}$  and  $\mathbf{B}$  must both verify property (a) of Lemma 1. If  $\Delta(\mathbf{M}) = 0$  then (8) implies that  $\mathbf{B}$  satisfies property (a) of Lemma 1 only if  $|\mathbf{B}| = 0$  or  $|\mathbf{B}| = 1$  while  $\mathbf{A}$  satisfies that property only if  $|\mathbf{A}| = 0$  or  $|\mathbf{A}| = 1$  or, because of Corollary 1,  $r \leq |\mathbf{A}| \leq (r + 1)^L - r$  or  $|\mathbf{A}| = (r + 1)^L - 1$  or  $|\mathbf{A}| = (r + 1)^L$ . From (8) it follows that  $|\mathbf{B}| = 0$  if and only if  $j = 0$  and  $|\mathbf{B}| = 1$  if and only if  $j = r - 1$ . Moreover, if  $j = r - 1$  then  $\mathbf{B} = \mathbf{0}^{(L+1)}$  and as a consequence,  $\mathbf{0}^L \not\subseteq \mathbf{A}$ . Hence, because of Corollary 1, one has that if  $j = r - 1$  then  $|\mathbf{A}| = 0$  or  $r \leq |\mathbf{A}| \leq (r + 1)^L - r - 1$  or  $|\mathbf{A}| = (r + 1)^L - 1$ . From the derived constraints on  $|\mathbf{A}|$  and  $|\mathbf{B}|$  and from Lemma 3 we get

$$\Delta(n) = 0 \iff \begin{cases} j = 0 & \text{and } (k = 0 \text{ or } k = 1 \text{ or } r \leq k \leq (r + 1)^L - r \\ & \text{or } k = (r + 1)^L - 1), \\ j = r - 1 & \text{and } (k = 0 \text{ or } r \leq k \leq (r + 1)^L - r - 1 \\ & \text{or } k = (r + 1)^L - 1). \end{cases} \quad (9)$$

For some of the remaining values of  $n$  we are able to show that  $\Delta(n) > 1$ . We consider the following four cases:

*Case 1:*  $k = 0$  and  $1 \leq j \leq r - 2$ . Let  $\mathbf{M}$  be as in (6) and suppose  $\Delta(\mathbf{M}) = \Delta(n)$ . If all non-empty entries of row  $L + 1$  of  $\mathbf{M}$  were 0, we could remove such entries and hence obtain an admissible table  $\mathbf{M}'$  with  $\Delta(\mathbf{M}') = W(\mathbf{M}') - H(n) < \Delta(n)$ . Therefore, there are non-empty entries of row  $L + 1$  which are different from 0. Condition (a) of Lemma 1 implies that these are at least  $r$ . Hence, one has  $n - |\mathbf{A}| \geq r$ , which using (7) gives  $\Delta(n) \geq r - j - 1$ .

Case 2:  $2(r+1) \leq n \leq r(r+1)$ . The lower bound for these values of  $n$  follows from by the lower bound for the sequential case [5].

Case 3:  $2 \leq j \leq r-2$ ,  $2r-j \leq k \leq 2r-2$  and  $L \leq r-j-3$ . We know from (9) that  $\Delta(\mathbf{M}) \geq 1$ . We show now that no admissible table exists with  $\Delta(\mathbf{M}) = 1$ .

**Fact 1.** Let  $2 \leq j \leq r-2$ ,  $2r-j \leq k \leq 2r-2$  and  $L \leq r-j-3$ . No admissible table

$$\mathbf{M} = \left[ \mathbf{A} \left| \mathbf{M}_{L+1} - \begin{bmatrix} \mathbf{A} \\ \mathbf{0}^{(|\mathbf{A}|)} \end{bmatrix} - \cdots - \begin{bmatrix} \mathbf{A} \\ \mathbf{r}^{(|\mathbf{A}|)} \end{bmatrix} - \mathbf{B} \right] \quad (10)$$

exists with  $|\mathbf{A}| = (r+1)^L - (k+2)$  and  $|\mathbf{B}| = 2r-j$ .

The (rather lengthy) proof is given in Appendix B.

Using Fact 1 and (7) we get that each admissible table  $\mathbf{M}$  has  $\Delta(\mathbf{M}) \geq 2$ , that is  $\Delta(n) \geq 2$ .

Case 4:  $k \geq (r+1)^L - r + 1 - \alpha(j)$ . Let  $\mathbf{M}$  be as in (6) and suppose  $\Delta(\mathbf{M}) = \Delta(n)$ . From (7), we have  $|\mathbf{A}| = (r+1)^L - k - \alpha(j) - \Delta(n)$ . We show that  $\Delta(n) \geq (r+1)^L - k - 1 - \alpha(j)$  by proving that  $|\mathbf{A}| \leq 1$ . If  $|\mathbf{A}| > 1$ , there exists a column  $\mathbf{c}$  of  $|\mathbf{A}|$  with  $\mathbf{c} \neq \mathbf{0}^{(L)}$ . Let  $\mathbf{M}'$  be the table obtained from  $\mathbf{M}$  by filling the  $|\mathbf{A}|$  empty entries in the  $(L+1)$ th row of  $\mathbf{M}$  with 0's. It is obvious that  $\mathbf{M}'$  is admissible and

$$\mathbf{M}' \subseteq \left[ \mathbf{M}_{L+1} - \begin{bmatrix} \mathbf{c} \\ 1 \end{bmatrix} \left| \cdots \right| \begin{bmatrix} \mathbf{c} \\ r \end{bmatrix} \right].$$

The admissibility of  $\mathbf{M}'$  implies that there exist  $t$  columns  $\mathbf{y}_1, \dots, \mathbf{y}_t$  such that

$$\mathbf{H} = \left[ \begin{bmatrix} \mathbf{c} \\ 1 \end{bmatrix} \left| \cdots \right| \begin{bmatrix} \mathbf{c} \\ r \end{bmatrix} \left| \mathbf{y}_1 \right| \cdots \left| \mathbf{y}_t \right] \right]$$

is admissible with  $\mathbf{M}' = [\mathbf{M}_{L+1} - \mathbf{H}]$ . Let  $c_i$  be a non-zero entry of  $\mathbf{c}$ . The  $i$ th row of  $\mathbf{H}$  satisfies property (a) of Lemma 1 only if the  $i$ th entries of  $\mathbf{y}_1, \dots, \mathbf{y}_t$  contain  $r$  occurrences of  $1, \dots, c_i - 1, c_i + 1, \dots, r$ . Therefore, one has that  $t \geq r(r-1)$  and as a consequence it results  $|\mathbf{M}'| = |\mathbf{M}| = n \leq (r+1)^{L+1} - r^2$ , that is  $k < (r+1)^L - r + 1 - \alpha(j)$ .  $\square$

## 4. Conclusion

The lowers bounds of Lemma 5 match the corresponding upper bounds of Lemma 3 except when  $(r+1)^{L+1} - r + 1 \leq n \leq (r+1)^{L+1} - 2$  for  $L \geq 2$  and  $(r+1)^L + r + 1 \leq n \leq (r+1)^L + r^2 - r - 1$  for  $L \geq 2$ . In those cases the differences between the upper and lower bounds to  $\bar{L}_{\text{pre}}(n)$  range between  $1/n$  and  $(r-1)/n$ . An interesting open problem is to reduce the gap between these lower and the upper bounds.

## Appendix A

Given a column vector  $x = (x_1, \dots, x_\ell)$ , with entries on  $\{0, 1, \dots, r\}$  and an integer  $0 \leq i \leq r$ , denote by  $y = x \oplus i$  the vector  $y = (y_1, \dots, y_\ell)$  with

$$y_j = \begin{cases} 0 & \text{if } x_j = 0, \\ x_j + i & \text{if } x_j \neq 0 \text{ and } x_j + i \leq r, \\ x_j + i \pmod{r} & \text{if } x_j \neq 0 \text{ and } x_j + i > r. \end{cases}$$

For each  $x \neq \mathbf{0}^\ell$ , let  $C(x)$  be the  $(\ell \times r)$  admissible table whose columns are  $x, x \oplus 1, \dots, x \oplus (r-1)$ . Call  $C(x)$  and  $C(y)$  disjoint if they do not have common columns, i.e.  $y \neq x \oplus i$  for each  $i$ . Let  $M_\ell$  denote the table whose columns are all vectors of length  $\ell$  on  $\{0, 1, \dots, r\}$ . It is easy to see that it is possible to choose  $((r+1)^\ell - 1)/r$  columns of  $M_\ell$ ,  $x_1, \dots, x_{((r+1)^\ell - 1)/r}$ , in such a way that  $C(x_1), \dots, C(x_{((r+1)^\ell - 1)/r})$  form a partition of  $[M_\ell - \mathbf{0}^\ell]$ .

**Proof of Lemma 2.** Write  $m$  as  $m = (r+1)^\ell - pr - t$  with  $0 \leq p < (r+1)^{\ell-1}$ ,  $\ell \geq 2$  and  $0 \leq t \leq r-1$ .

If  $t = 0$  and  $p \geq 2$  or if  $t > 0$  and  $p \geq 1$ , let

$$s = \begin{cases} r & \text{if } t = 0 \\ t & \text{if } t > 0 \end{cases}$$

and let  $B$  be the  $(\ell \times (r+s))$  table defined as follows:

$$B = \begin{cases} \begin{bmatrix} 1 & \dots & r & 0 \\ 1 & \dots & r & 0 \\ 1 & \dots & r & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & r & 0 \end{bmatrix} & \text{if } s=1, \\ \begin{bmatrix} 1 & 2 & \dots & s-1 & s & \dots & r & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & r-s+1 & r-s+2 & \dots & r & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & r-s+1 & r-s+2 & \dots & r & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & \dots & r-s+1 & r-s+2 & \dots & r & 0 \end{bmatrix} & \text{if } s > 1. \end{cases}$$

We choose

$$h = \begin{cases} p & \text{if } t = 0, \\ p-1 & \text{if } t > 0, \end{cases}$$

vectors  $x_1, \dots, x_h$  of  $M_\ell - \mathbf{0}^\ell$  such that the tables  $C(x_1), \dots, C(x_h)$ ,  $C(s, 1, 1, \dots, 1)$ ,  $C(1, 0, 0, \dots, 0)$  and  $C(0, r, r, \dots, r)$  are pairwise disjoint. Notice that  $C(x_1), \dots, C(x_h)$  do not contain columns of  $B$ . The desired table  $A$  can be obtained by deleting from  $M_\ell$  the columns of  $C(x_1), \dots, C(x_h)$  and, if  $h = p-1$ , the columns of  $B$ . If  $h > 0$  and  $t \neq 1$ , it is possible to choose  $x_1 = (1, 1, \dots, 1)$ , i.e.  $C(x_1) = [1^\ell \ 2^\ell \ \dots \ r^\ell]$ ; if  $p \geq 1$  and  $t = 1$ , the columns  $1^\ell 2^\ell \dots r^\ell$  belong to  $B$ . In either cases  $A$  does not



contain any of the columns  $\mathbf{1}^{\ell} \dots \mathbf{r}^{\ell}$ . We can easily convince ourselves that these two are the only cases when an  $\ell \times m$  admissible table can be built in such a way that it does not contain any of the columns  $\mathbf{1}^{\ell} \dots \mathbf{r}^{\ell}$ .

If  $p = 0$  and  $t = 0$ , then we set  $A = M_{\ell}$ . If  $p = 1$  and  $t = 0$ , i.e.,  $m = (r+1)^{\ell} - r$ , any admissible table is obtained by deleting from  $M_{\ell}$  the columns of  $C(x)$  for some column  $x \neq \mathbf{0}^{\ell}$ .

If  $p = 0$  and  $t = 1$ , i.e.,  $m = (r+1)^{\ell} - 1$ , the only admissible table is obtained by removing the all-zero column from  $M_{\ell}$ .

If  $p = 0$  and  $2 \leq t \leq r-1$ , i.e.,  $m = (r+1)^{\ell} - t$ , there exists no admissible table of size  $\ell \times m$ . Anyhow, it is easy to verify the existence of an admissible table  $A$  of size  $(\ell+1) \times m$ . Moreover, such a table  $A$  can be constructed in such a way that it does not contain the all-zero column.

Notice that the table  $A$  has been built in such a way that it contains the all-zero column only if  $m = (r+1)^{\ell}$  or  $m = (r+1)^{\ell} - r$ . For these two values of  $m$  it is impossible to build an  $\ell \times m$  admissible table  $A$  such that  $A$  does not contain the all-zero column. Notice also that if  $m \leq (r+1)^{\ell} - 2r$  and  $m = (r+1)^{\ell} - r - 1$  then  $A$  does not contain any of the columns  $\mathbf{0}^{\ell} \mathbf{1}^{\ell} \dots \mathbf{r}^{\ell}$ . For all the remaining values of  $m$  it is not possible to build an  $\ell \times m$  admissible table which does not contain any of the columns  $\mathbf{0}^{\ell} \mathbf{1}^{\ell} \dots \mathbf{r}^{\ell}$ .

## Appendix B

It is useful to introduce the following notation: given a table  $T$ , we denote with  $T^*$  the table containing  $r$  occurrences of each column of  $T$ , e.g. if  $r = 2$  and  $T = [c_1 \ c_2 \ \dots \ c_m]$  then  $T^* = [c_1 \ c_1 \ c_2 \ c_2 \ \dots \ c_m \ c_m]$ , and with  $T^i$  the table consisting of the first  $i$  rows of  $T$ .

**Proof of Fact 1.** Given  $2r-j \leq k \leq 2r-2$ ,  $2 \leq j \leq r-2$  and  $L \leq r-j-3$ , let us suppose that there exists an admissible table

$$\begin{aligned} M &= \left[ A \mid M_{L+1} - \begin{bmatrix} A \\ \mathbf{0}^{(|A|)} \end{bmatrix} - \dots - \begin{bmatrix} A \\ \mathbf{r}^{(|A|)} \end{bmatrix} - B \right] \\ &= \left[ M_L - H \mid \begin{bmatrix} H \\ \mathbf{0}^{(|H|)} \end{bmatrix} - \dots - \begin{bmatrix} H \\ \mathbf{r}^{(|H|)} \end{bmatrix} - B \right] \end{aligned} \quad (\text{B.1})$$

with  $|A| = (r+1)^L - (k+2)$  and  $|B| = 2r-j$ . It is  $H = [M_L - A]$  and  $|H| = k+2$ . In the following, we abuse the term “admissible” to denote a table verifying property (a) of Lemma 1 (notice that  $M$ ,  $A$ ,  $B$  and  $H$  are all supposed to satisfy property (b) of Lemma 11).

Notice that from (B.1) we have

$$A \text{ is admissible} \iff H \text{ is admissible} \iff B^L \text{ is admissible.} \quad (\text{B.2})$$

We shall prove that  $A$  and  $B$  cannot be both admissible. Let us assume by contradiction that  $A$  and  $B$  are both admissible tables. From (B.1), we get that  $B^L$  contains only occurrences of columns belonging to  $H$  and, since  $r + 2 \leq |B| < 2r - 1$  then  $B^L$  has one of the two possible forms:

$$B^L = \begin{cases} [H' | 0^{(L \times (|B| - |H'|))}] & \text{with } H' \subset H \text{ and } [0^{(L)}] \subset H, \\ H' & \text{with } H' \subseteq H \end{cases}$$

where  $H'$  is some admissible table with  $[0^{(L)}] \not\subseteq H'$ . Therefore, in the former case we must have  $|H'| \geq r$  while in the latter one  $|H'| = |B| \geq r + 2$ . Being  $|H| \leq 2r$ , we get  $|H - H' - 0^L| \leq r - 1$  in the former case and  $|H - H'| \leq r - 2$  in the latter case. As a consequence  $[H - H']$  cannot be admissible. Since  $H$  is obtained by joining the columns of an admissible table with those of a non-admissible one, it may not be admissible. This obviously contradicts our assumption that  $A = [M_L - H]$  is admissible.

Since  $A$  and  $B$  cannot be both admissible then (B.2) implies that  $A$  and  $B$  are both non-admissible. Assume then that  $A$  and  $B$  are not admissible and let  $\ell \leq L$  be the number of rows of  $A$  which do not satisfy property (a). Without loss of generality we assume that the first  $\ell$  rows of  $A$  do not satisfy (a). Let us observe that for every  $1 \leq t \leq \ell$  there exists exactly one integer  $a_t \in \{1, 2, \dots, r\}$  such that the  $t$ th row of  $A$  contains the same number of occurrences of all integers in  $\{1, 2, \dots, r\} - \{a_t\}$  and one less occurrence of  $a_t$ . Our observation follows from the fact that  $M$  is admissible only if for each less occurrence of a certain integer in the  $t$ th row of  $A'$  there are  $r$  more occurrences of that integer in the  $t$ th row of  $B'$ . Since  $|B| < 2r$  then there is at most one non-zero integer, which we have denoted with  $a_t$ , which occurs  $r$  times in the  $t$ th row of  $B'$ . The admissibility of  $M$  implies that for every  $t = 1, 2, \dots, \ell$  the  $t$ th row of  $B$  contains  $r$  more occurrences of  $a_t$ , i.e., it contains  $r$  occurrences of  $a_t$  and  $r - j$  occurrences of 0. Hence each column  $c = (c_1, \dots, c_L)$  of  $B$  has  $c_t \in \{a_t, 0\}$ , for  $t = 1, \dots, \ell$ .

We shall prove that  $B'$  contains at most  $\ell + 2$  distinct columns. Denote by  $\mathcal{C}_t$  the subset of distinct columns of  $B^L$  having  $a_t$  in the  $t$ th entry, for  $t = 1, \dots, \ell$ . We prove that  $|\mathcal{C}_t| \leq 2$ , for  $t = 1, \dots, \ell$ . Suppose on the contrary  $|\mathcal{C}_t| \geq 3$ . Since  $B^L \subset H^*$ , these columns of  $B^L$  belong to  $H$ , too. As the  $t$ th row of  $H$  contains exactly one more occurrence of  $a_t$  than those of each  $a \in \{1, \dots, r\} - \{a_t\}$ , then it must contain at least two occurrences of each  $a \neq a_t$ , that is  $|H| \geq 2(r - 1) + 3 = 2r + 1$ , contradicting our assumption. We show now that for every  $s \neq t$ ,  $1 \leq s \leq \ell$ , at least one of the columns in  $\mathcal{C}_t$  has the  $s$ th entry equal to  $a_s$ . Assume by contradiction that no column of  $\mathcal{C}_t$  has the  $s$ th entry equal to  $a_s$  for some integer  $s$  such that  $1 \leq s \leq \ell$  and  $s \neq t$ . Then,  $r$  columns of  $B$  would be necessary to establish admissibility on the  $t$ th row of  $M$  and other  $r$  columns of  $B$  to establish admissibility on the  $s$ th row of  $M$  and it would absurdly result  $|B| \geq 2r$ . Therefore,  $\mathcal{C}_t \cap \mathcal{C}_s \neq \emptyset$  for every  $t \neq s$ . Recalling that  $|\mathcal{C}_t| \leq 2$  for every  $t$ , we get that  $|\bigcup_{i=1}^{\ell} \mathcal{C}_i| \leq \ell + 1$ . As  $B'$  contains only the columns of  $\mathcal{C}_1, \dots, \mathcal{C}_{\ell}$  and the all-zero columns, the number of distinct columns of  $B'$  is at most  $\ell + 2$ .

We finally prove that for each  $p > \ell$ , each entry in the  $p$ th row of  $\mathbf{B}^L$  is 0. Suppose by contradiction that there is a non zero integer in the  $p$ th row of  $\mathbf{B}^L$ , for some  $p > \ell$ . Since this row has to verify property (a) of Lemma 1 then it contains one occurrence of the integers  $1, 2, \dots, r$  and  $r - j$  occurrences of 0. Let  $\mathbf{v}_i$  denote the column of  $\mathbf{B}^L$  having the  $p$ th entry equal to  $i$ , for  $i = 1, \dots, r$ . It is  $|\mathcal{C}_t \cap \{\mathbf{v}_1, \dots, \mathbf{v}_r\}| \leq |\mathcal{C}_t| \leq 2$ , for every  $t \leq \ell$ . It cannot be  $|\mathcal{C}_t \cap \{\mathbf{v}_1, \dots, \mathbf{v}_r\}| = |\mathcal{C}_t| = 2$  since  $\mathbf{B}^L$  must contain at least  $\lceil r/2 \rceil$  occurrences of one of the columns in  $\mathcal{C}_t$  and exactly one occurrence of each  $\mathbf{v}_i$ . Suppose then  $|\mathcal{C}_t \cap \{\mathbf{v}_1, \dots, \mathbf{v}_r\}| \leq 1$ . In this case, at most one column of  $\mathcal{C}_t$  coincides with some  $\mathbf{v}_i$ . Hence at least  $r - 1$  columns of  $\mathbf{B}^L$  belong to  $\mathcal{C}_t \setminus \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$  and as a consequence it absurdly results  $2r - j = |\mathbf{B}| \geq 2r - 1$ . Therefore, the last  $(L - \ell)$  rows of  $\mathbf{B}^L$  are all-zero vectors and  $\mathbf{B}_L$  contains at most  $\ell + 2 \leq L + 2 < r - j$  pairwise distinct columns. Hence, there are at least  $r + 1$  columns of  $\mathbf{B}^L$  such that each of them is identical to some other column of  $\mathbf{B}^L$ . Since  $\mathbf{B}$  must satisfy property (b) of Lemma 1 then the columns of  $\mathbf{B}$  must be pairwise distinct. The last row of  $\mathbf{B}$  has to satisfy property (a) of Lemma 1 and therefore it is either an all-zero vector or contains one occurrence of the integers  $1, \dots, r$  and  $r - j$  zeros. For that reason the last row of  $\mathbf{B}$  cannot be built in such a way that the columns of  $\mathbf{B}$  are pairwise distinct.

From our discussion, it follows that there is no couple of matrices  $\mathbf{B}$  and  $\mathbf{H}$  satisfying our instances proving that no admissible table  $\mathbf{M}$  exists with the given parameters.

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